SYMMETRIC MONOIDAL PREORDERS AND APPLICATIONS TO FINITE SPACES

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ABSTRACT. This paper aims to offer a glimpse into the intersection of mathematics between preorders, category theory, and finite topological spaces. In particular, we will showcase interesting interactions between Galois connections, closed symmetric monoidal preorders, homotopy equivalence, and higher-dimensional topologies.

CONTENTS

1. INTRODUCTION

This paper is in large part motivated by one particular problem. A problem which one could characterize as being *multi-dimensional* both literally and figuratively. Through the study of finite topological spaces, many will learn of the *one-to-one correspondence* between the Alexandroff topologies and the preorders on any given set. Thus, if we were to consider all of the possible Alexandroff topologies on a given set and assign an ordering to this set of topologies, we would be able to obtain a new topology; that is, a topology with topologies as its points. It is this particular topology that this paper intends to study and provide preliminary results for, with the hope that further study surrounding "higher-dimensional" topologies will be conducted.

In Section 2, we will study preorders and introduce the notion of a *Galois connection*, a process which we may describe as a *relaxed isomorphism* between preorders.

In Section 3, we will briefly glimpse into *higher-dimensions* by constructing a *preorder of preorders*. In particular, we will observe a Galois connection between the preorderings and relations on any set.

In Section 4, we will familiarize ourselves with a special type of preorder, one which possesses a symmetric monoidal structure. From this, we will introduce the concept of *enrichment*, a process one may describe as being able to *cross higher-dimensions*. We will also observe an interesting interaction between Galois connections and symmetric monoidal preorders which enrich in a closed manner.

In Section 5, we will introduce finite topological spaces and finally present our motivating problem; that is, the *topology of topologies*. We will then observe a profound interaction between Galois connections and homotopy theory and discuss its subsequent applications.

2. Preorders

For a moment, let us put mathematical abstraction to the side and shift our focus to something more physical and tangible. I invite you to recall any decisions which you have made recently. This could involve choosing to read this paper, going to the bathroom, or something less trivial such as selecting between similar food products at a grocery store. Having seen the title of this section, it may not come as a surprise when I claim that preorders are at play throughout these decisions. For instance, suppose that we are at a grocery store trying to choose one peanut butter jar out of a dozen to purchase. We may look at numerous factors when deciding this such as brand, taste, or cost. Regardless of our criteria, we will end up assigning value to each of the jars and *ordering* them as such which allows us to pinpoint the jar with the greatest value. Hence, we can see that an act as simple as choosing between food products possesses an underlying preorder structure.

Now that you are filled to the brim with enthusiasm towards the study of preorders, we can bring back our mathematical abstraction and focus back into the contents of this paper. In this section, we will explore various preorder structures, operators on them, and mappings between them. This will enable us to introduce a special case of mappings between preorders called a *Galois connection* which will eventually allow us to bridge our study of preorders to category theory and topology.

Definition 2.1. A *preorder relation* on a set X is a binary relation on X, here denoted with infix notation $\leq$, such that

(a) (*reflexivity*) $x \leq x$ for all $x \in X$, and
(b) (*transitivity*) if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in X$.

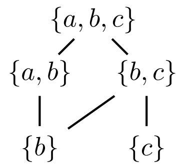
If $x \leq y$ and $y \leq x$, we write $x \cong y$ and say x and y are *equivalent*. We call a pair $(X, \leq)$ consisting of a set equipped with a preorder relation a *preorder*.

Definition 2.2. A preorder is a *partial order* if we additionally have that

(c) (*antisymmetry*) $x \cong y$ implies $x = y$.

We call a pair $(X, \leq)$ consisting of a set equipped with a partial order a *poset*.

Example 2.3. Consider the set $X = \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}, \{c\}\}$. An example of a partial order is given by ordering this set by subset inclusion; that is, $A \leq B$ if and only if $A \subseteq B$. A visual representation of this partial order is given by the following diagram in which $A \subseteq B$ if there is a line from A which goes up to B.



We will see shortly that this diagram is actually a special type of *graph* called a *Hasse diagram*.

Example 2.4. One may naturally wonder if there exists a preorder which is not a partial order. For this, we can consider a rather practical example. Imagine we have a set of people $X = \{Andrew, Bruce, Charlie, Daniel\}$. Given two people $A, B \in X$, we say that $A \leq B$ if B is taller than A. One can check that this ordering meets the reflexive and transitive requirements. Let us now suppose that Andrew and Bruce have the same height. It follows that $Andrew \leq Bruce$ and $Bruce \leq Andrew$. However, from this we cannot conclude that $Andrew = Bruce$. After all, while *Andrew* and *Bruce* may possess the same height, that does not imply that they are the same person. Hence, we have equipped this set with a preordering which is not a partial order.

Definition 2.5. A *Hasse diagram* is a directed graph which gives a representation of a preorder $(P, \leq)$. The elements of P are the vertices V, and the order $\leq$ is given by $v \leq w$ if and only if there is a path $v \to w$. For any vertex v, there is always a path $v \to v$, which satisfies reflexivity. The paths $u \to v$ and $v \to w$ can be concatenated to a path $u \to w$ satisfies transitivity. In our depictions of Hasse diagrams, we will only draw arrows between immediate elements; that is, for any two vertices $v, w \in V$, we will draw the path between them if and only if $v \to w$ is the only path between them. Furthermore, these paths will always have upwards vertical direction.

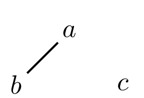
Theorem 2.6. *The above construction of the Hasse diagram gives a bijection from partial orders to partial order diagrams. Furthermore, there is an isomorphism of partial orders between two posets P and Q if and only if there is a graph isomorphism between the associated diagrams H_P and H_Q.*

Proof. For details on this theorem/proof see May [[2], Thm. 2.6.3.] □

Example 2.7. Consider the set $X = \{a, b, c\}$ equipped with the following preordering (omitting reflexive and transitive laws):

$$b \leq a.$$

This can be represented with the following Hasse diagram:



Having seen various examples of preorders with examination of their individual structures, we now wish to study how we can get from one preorder to another; that is, the mappings between preorders. We now define a special type of map which *preserves structure* between preorders.

Definition 2.8. A *monotone map* between preorders $(A, \leq_A)$ and $(B, \leq_B)$ is a function $f : A \to B$ such that, for all elements $x, y \in A$, if $x \leq_A y$ then $f(x) \leq_B f(y)$.

Remark 2.9. In other contexts, a monotone map may be referred to as an *order-preserving map*. It is also worth noting that in the special case of finite topological spaces, monotone maps are equivalent to *continuous maps*. This scenario is something we will later encounter and formalize.

Example 2.10. Recall our preorder from (2.4). Let Y be a set containing the biological parents of those contained in X. We define an ordering $\leq_Y$ on Y analogously to the ordering on X; that is, an ordering by height. We now define a function $f : X \to Y$ which maps a child to their father. For the purposes of this example, let us assume that all children are $\frac{3}{4}$ the height of their father. It follows then that given $A, B \in X$ where $A \leq_X B$ we have that $f(A) \leq_Y f(B)$. As such, f is a monotone function.

We will now define a pair of operators on preorders. These operators, called *meets* and *joins*, allow us formulate a particular structure on preorders as we will see shortly.

Definition 2.11. Let $(P, \leq)$ be a preorder, and let $A \subset P$ be a subset. We say that an element $p \in P$ is a *meet* of A if

(a) for all $a \in A$, we have $p \leq a$, and
(b) for all $q \in P$ such that $q \leq a$ for all $a \in A$, we have that $q \leq p$.

We write $p = \bigwedge A$, $p = \bigwedge_{a \in A} a$, or, if the dummy variable a is clear from context, just $p = \bigwedge_A a$. If A consists of just two elements, say $A = \{a, b\}$, we can write $\bigwedge A$ simply by $a \wedge b$.
Similarly, we say that p is a *join* of A if

(a) for all $a \in A$ we have $a \leq p$, and
(b) for all $q \in P$ such that $a \leq q$ for all $a \in A$, we have that $p \leq q$.

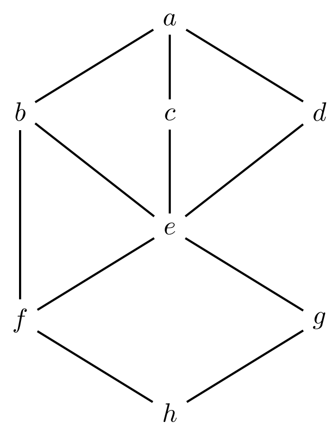
We write $p = \bigvee A$ or $p = \bigvee_{a \in A} a$, or when $A = \{a, b\}$ we may simply write $p = a \vee b$.

Remark 2.12. One can think of *meets* and *joins* as being equivalent to the *infimum* and *supremum* of a set. That is, given $A \subset X$ where X is a set equipped with a preordering, we have the following:

$$\bigwedge A \cong \inf A \text{ and } \bigvee A \cong \sup A$$

where $\inf A, \sup A \in X$. However, it is worth noting that unlike the infimum and supremum, meets and joins are not necessarily *unique* as we will see shortly.

Example 2.13. Consider the following preorder $(X, \leq)$:



(i) Consider the preorder X in its entirety. We have that

$$\bigwedge X = h \text{ and } \bigvee X = a.$$

(ii) Suppose that we remove h from X. Does $\bigwedge X$ still exist?

(iii) Consider the subset $A = \{b, c, e\}$. We have that

$$\bigwedge A = e \text{ and } \bigvee A = a.$$

(iv) Consider the subset $A = \{b, c\}$. Suppose that we remove e from X while retaining the transitive relations stemming from it. We have that $\bigwedge X = \{f, g\}$.

(v) Given any $x, y, z \in X$, can we prove that the following equality holds?

$$(x \vee y) \vee z = x \vee (y \vee z)$$

(vi) Given any $x_1, x_2, y_1, y_2 \in X$ such that $x_1 \leq y_1$ and $x_2 \leq y_2$, can we prove that the following inequality holds?

$$x_1 \vee x_2 \leq y_1 \vee y_2$$

(vii) Can we generalize parts (v) and (vi) to any preorder?

Definition 2.14. A *Galois connection* between preorders P and Q is a pair of monotone maps $f : P \to Q$ and $g : Q \to P$ such that for all $p \in P$ and $q \in Q$

$$f(p) \leq q \text{ if and only if } p \leq g(q).$$

We call f the *left adjoint* and g the *right adjoint* of the Galois connection.

One can think of a Galois connection as a pair of maps which preserve the ordering of points in *two different contexts simultaneously.* From this, it might seem that a Galois connection is akin to a *relaxed isomorphism.* In fact, we find that all preorder isomorphisms are indeed Galois connections while not all Galois connections are preorder isomorphisms.

Proposition 2.15. *Suppose that $f : P \to Q$ and $g : Q \to P$ are monotone maps. The following are equivalent*

(a) f and g form a Galois connection where f is left adjoint to g,

(b) for every $p \in P$ and $q \in Q$ we have

$$p \leq g(f(p)) \text{ and } f(g(q)) \leq q$$

Proof. We will first show that statement (a) implies statement (b).

Let $p \in P$. By reflexivity, we have the following inequality:

$$p \leq p.$$

Since f is a monotone map, it follows that

$$f(p) \leq f(p).$$

Since f maps from P to Q, we have that $f(p) \in Q$. Hence, since f and g form a Galois connection where f is left adjoint to g, it follows that

$$f(p) \leq f(p) \text{ if and only if } p \leq g(f(p)).$$

Therefore, since $f(p) \leq f(p)$, we have our result

$$p \leq g(f(p)).$$

as desired. An analogous proof follows to show that $f(g(q)) \leq q$.

We will now show that statement (b) implies statement (a).

We want to show that f and g form a Galois connection where f is left adjoint to g. That is,

$$f(p) \leq q \text{ if and only if } p \leq g(q).$$

Suppose that $f(p) \leq q$. Since g is a monotone map, it follows that $g(f(p)) \leq g(q)$. From our supposition (b), we have that $p \leq g(f(p))$ and so by transitivity:

$$p \leq g(q).$$

An analogous proof follows to show that $p \leq g(q)$ implies $f(p) \leq q$. □

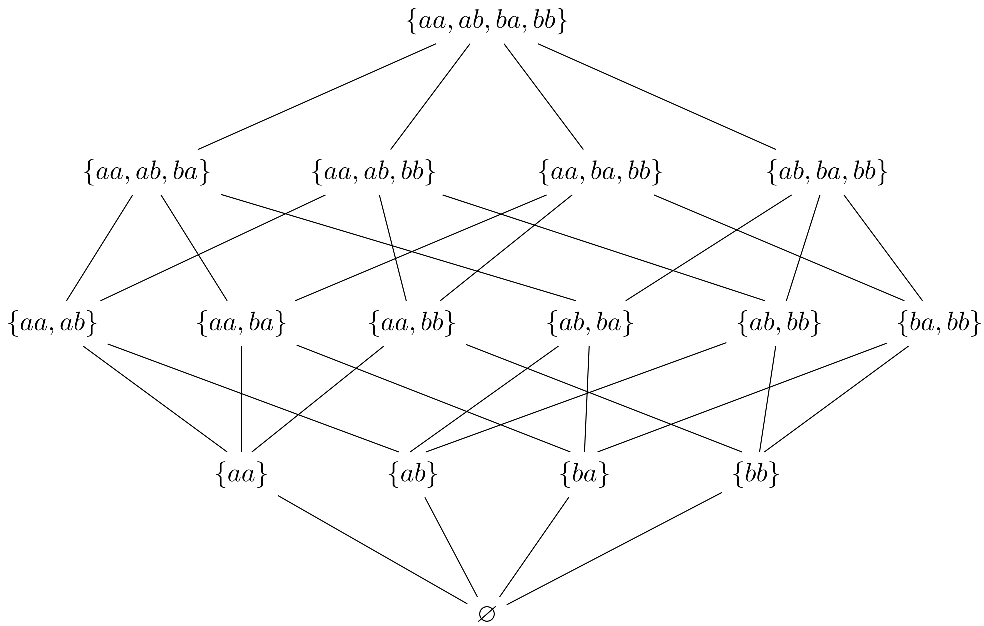
From this result, we have acquired an equivalent definition of a Galois connection. For readers who are already familiar with homotopy theory, one may realize that this characterization of a Galois connection looks almost identical to that of a homotopy equivalence. We will later see that this is indeed the case.

3. Level Shifting

Through the study of mathematics, one may be inclined to ponder *higher dimensional* questions. In topology, this is exhibited through the notion of a homotopy between homotopies or a topology generated by topologies, something which we will discuss later. We will informally refer to this phenomenon as *level shifting* (as coined by Fong and Spivak [[1], 1.4.5.]) and proceed to study a particular example of level shifting pertaining to the preorder of preorders.

Definition 3.1. Given any set S, we define $\mathbf{Rel}(S)$ to be the set of all possible binary relations on S. An element $R \in \mathbf{Rel}(S)$ is a subset $R \subset S \times S$ with binary relations represented by ordered pairs (x, y) as its elements. We can assign an ordering to $\mathbf{Rel}(S)$ by subset inclusion $R \subseteq R'$; that is, for every $x, y \in S$, if $R(x, y)$ then $R'(x, y)$.

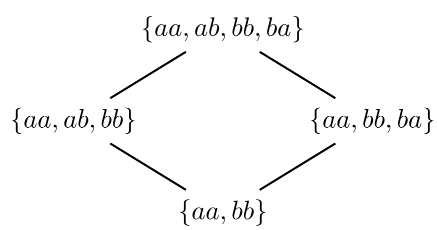
Example 3.2. Here we denote the ordered pair (a, b) as ab for sake of brevity. See the Hasse diagram for $\mathbf{Rel}(\{a, b\})$:



Definition 3.3. Given any set S, we define $\mathbf{Pos}(S)$ to be the set of all possible preorder relations on S with identical structure to that of $\mathbf{Rel}(S)$.

By equipping $\mathbf{Pos}(S)$ with an ordering by subset inclusion $\subseteq$, one can check that this forms a preorder structure on $\mathbf{Pos}(S)$. Hence, we have a preorder structure on a set of preorders. As such, this is precisely a *level shift*.

Example 3.4. See the Hasse diagram for $\mathbf{Pos}(\{a, b\})$:

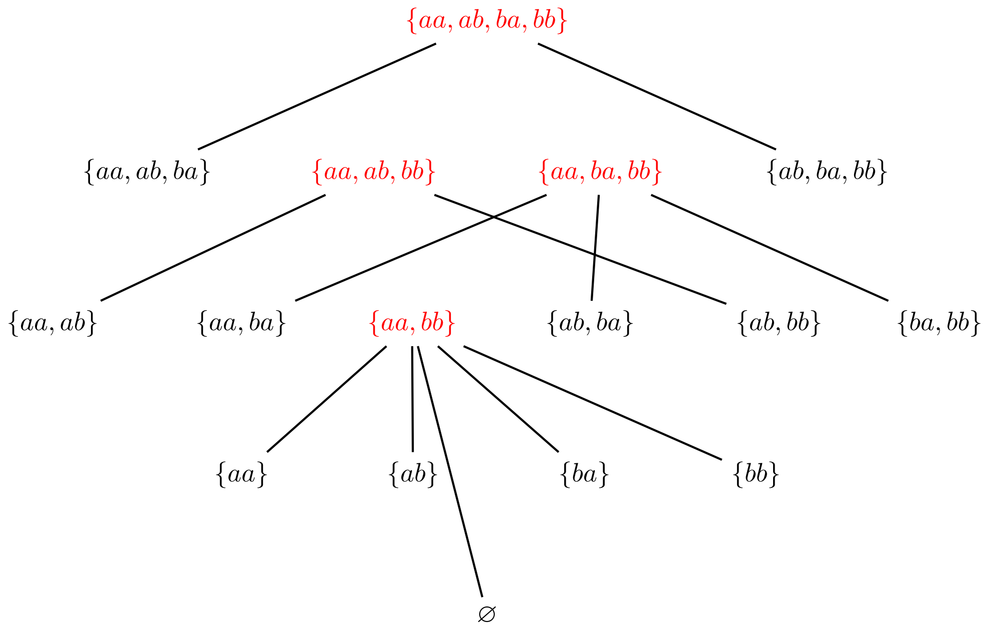


Proposition 3.5. *Notice that every preorder is in fact a relation and so there is an inclusion map* $g : \mathbf{Pos}(S) \to \mathbf{Rel}(S)$. *Let* $f : \mathbf{Rel}(S) \to \mathbf{Pos}(S)$ *be given by taking any relation* R*, writing in infix notation using* $\leq$*, and taking the reflexive and transitive closure; that is, adding* $s \leq s$ *for every* $s \in S$ *and adding* $s \leq u$ *whenever* $s \leq t$ *and* $t \leq u$.

There is a Galois connection between $\mathbf{Pos}(S)$ *and* $\mathbf{Rel}(S)$ *given by* f *as the left adjoint and* g *as the right adjoint.*

A proof will not be provided for this proposition. Instead, we will work through an example with a two point set in belief that this exercise will be more informative than the abstract proof. Readers are encouraged to generalize results from this example and formulate their own proof.

Example 3.6. Let $S = \{a, b\}$. The following diagram illustrates the mappings of the left adjoint f which converts relations to preorderings. Relations are colored black while preorderings are colored red.



From seeing the mappings of the left adjoint f, one will notice that f is monotone. We can conceptually explain this. Suppose we are given two relations R, R' such that $R \leq R'$. Since $R \leq R'$, one can say that all of the *information* contained within R was also contained in R'. The process of taking the reflexive and transitive closure on a relation can be described as *building up* from existing information i.e. forming connection from existing ones. Therefore, arguing by contradiction, how can it be the case that $f(R) \not\leq f(R')$? After all, this would imply that $f(R)$ contained an element not present in $f(R')$ despite the fact that R did not contain any more information than R' to begin with.

4. Symmetric Monoidal Preorders

In this section we will focus on a special type of preorder called a *symmetric monoidal preorder*. With the unique structure of symmetric monoidal preorders, we will be able to study the process of *enrichment* and observe Galois connections within a special subset of symmetric monoidal preorders, namely, ones which interact with enrichment in a *closed* manner.

Definition 4.1. A *symmetrical monoidal structure* on a preorder $(X, \leq)$ consists of two constituents:

(i) an element $I \in X$, called the *monoidal unit*, and
(ii) a function $\otimes : X \times X \to X$, called the *monoidal product*.

These constituents must satisfy the following properties, where we write $\otimes(x_1, x_2) = x_1 \otimes x_2$:

(a) (*monotonicity*) for all $x_1, x_2, y_1, y_2 \in X$, if $x_1 \leq y_1$ and $x_2 \leq y_2$, then $x_1 \otimes x_2 \leq y_1 \otimes y_2$,
(b) (*unitality*) for all $x \in X$, the equations $I \otimes X = x$ and $x \otimes I = x$ hold,
(c) (*associativity*) for all $x, y, z \in X$, the equation $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ holds, and
(d) (*symmetry*) for all $x, y \in X$, the equation $x \otimes y = y \otimes x$ holds.

A preorder equipped with a symmetrical monoidal structure, $(X, \leq, I, \otimes)$, is called a *symmetric monoidal preorder*.

Example 4.2. Consider the natural numbers including 0: $\mathbb{N} \cup \{0\}$ equipped with its standard preordering $\leq$. We choose our monoidal product and unit to be $+$, standard addition, and 0, respectively.

Example 4.3. Consider the booleans: $\mathbb{B} = \{\texttt{false}, \texttt{true}\}$ with the preordering $\leq$ such that $\texttt{false} \leq \texttt{true}$. We choose our monoidal product to be $\wedge$, the `and` operator and our monoidal unit to be `true`. We will call this symmetric monoidal preorder:

$$\mathbf{Bool} := (\mathbb{B}, \leq, \texttt{true}, \wedge).$$

One can check that equipping the booleans with $\vee$ (OR) as the monoidal product and `false` as the monoidal unit also satisfies the properties of a symmetric monoidal structure.

Example 4.4. Let $[0, \infty]$ denote the set of non-negative real numbers together with ∞. Consider the preorder $([0, \infty], \geq)$, with the usual notion of $\geq$, where of course $\infty \geq x$ for all $x \in [0, \infty]$.

There is a monoidal structure on this preorder, where the monoidal unit is 0 and the monoidal product is $+$. In particular, $x + \infty = \infty$ for every $x \in [0, \infty]$. Let us call this monoidal preorder:

$$\mathbf{Cost} := ([0, \infty], \geq, 0, +).$$

Example 4.5. Given any set X, recall the ordering on $\mathbf{Pos}(X)$ given by subset inclusion (3.3). We have that $(\mathbf{Pos}(X), \subseteq, S, \cap)$ is a symmetric monoidal preorder where S is the maximal preorder on X; that is, for every $x, y \in X$ we have that $S(x, y)$.

Now we will explore a symmetric monoidal structure using joins on $\mathbf{Pos}(X)$ where X is a finite set. We will use this particular symmetric monoidal preorder in later sections.

Proposition 4.6. *Let X be any finite set. There exists a symmetric monoidal structure on $(\mathbf{Pos}(X), \subseteq)$, given by I and $\vee$ where $\vee$ is the join operator and $I = \bigwedge \mathbf{Pos}(X)$ is the unique meet of $\mathbf{Pos}(X)$.*

Proof. We first wish to show that $\bigwedge \mathbf{Pos}(X)$ exists and is unique. Let us define a particular preordering on X:

$$I := \{(x, x) \text{ for all } x \in X\}.$$

Now let us check that I is a preordering on X. Simply from construction, *reflexivity* is satisfied. Since I contains only reflexive binary relations, *transitivity* is also satisfied. Hence, I is a preordering on X, and so $I \in \mathbf{Pos}(X)$.

We now claim that $\bigwedge \mathbf{Pos}(X) = I$. Since every possible preordering on X must satisfy reflexivity, it follows that for every $R \in \mathbf{Pos}(X)$, we have that $I \subseteq R$. This satisfies (a) from (2.12). Let $Q \in \mathbf{Pos}(X)$ such that $Q \subseteq R$ for all $R \in \mathbf{Pos}(X)$. Since $I \in \mathbf{Pos}(X)$, we have that $Q \subseteq I$. This satisfies (b) from (2.12). Therefore, $I = \bigwedge \mathbf{Pos}(X)$.

We now claim that $\bigwedge \mathbf{Pos}(X) = I$ is unique. Suppose by contradiction that there exists another meet of $\mathbf{Pos}(X)$, denoted J such that $J \neq I$. Since J and I are meets of $\mathbf{Pos}(X)$ and $J, I \in \mathbf{Pos}(X)$, we have that $J \subseteq I$ and $I \subseteq J$. Therefore, $I = J$ which forms a contradiction.

We now wish to show that I and $\vee$ form a symmetric monoidal structure on $(\mathbf{Pos}(X), \subseteq)$ where we take I as our *monoidal unit* and $\vee$, the join operator, to be our *monoidal product*. We will show that each property is indeed satisfied (Note: we use $\leq$ and $\subseteq$ interchangeably here).

(a) Let $P_1, P_2, Q_1, Q_2 \in \mathbf{Pos}(X)$ such that $P_1 \leq Q_1$ and $P_2 \leq Q_2$. Let $P = P_1 \vee P_2$ and let $Q = Q_1 \vee Q_2$. It follows that $P \geq P_1, P_2$ and $Q \geq Q_1, Q_2$. By transitivity, it also follows that $Q \geq P_1, P_2$. Therefore, by (b) from (2.12), we have that $P \leq Q$ as desired.
(b) Let $R \in \mathbf{Pos}(X)$. From (d), we have that $\vee$ is commutative. Since $I \leq R$, we have that $I \vee R = R = R \vee I$ as desired.
(c) Let $P, R, Q \in \mathbf{Pos}(X)$. Let $A = (P \vee R) \vee Q$ and let $B = P \vee (R \vee Q)$. We have that $A \geq P, R, Q$ and $B \geq P, R, Q$. Therefore, by (b) from (2.12), it follows that $A \subseteq B$ and $B \subseteq A$; that is, $A = B$ as desired.
(d) Let $P, Q \in \mathbf{Pos}(X)$. Since $\vee$ operates on sets and is well-defined, it follows that $P \vee Q = \bigvee\{P, Q\} = Q \vee P$ as desired.

Therefore, $(\mathbf{Pos}(X), \subseteq, I, \vee)$ is a symmetric monoidal preorder. □

The importance of the following proposition will be realized later once we discuss finite topological spaces.

Proposition 4.7. *Suppose $\mathcal{X} = (X, \leq)$ is a preorder and $X^{op} = (X, \geq)$ is its opposite. If $(X, \leq, I, \otimes)$ is a symmetric monoidal preorder then so is its opposite, $(X, \geq, I, \otimes)$.*

From this next definition, the significance of symmetric monoidal preorders may become apparent. We will now introduce a concept called *enrichment* in which we are able to construct a new mathematical structure by using a symmetric monoidal preorder as its *base*.

Definition 4.8. Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a symmetrical monoidal preorder. A $\mathcal{V}$-*category* $\mathcal{X}$ consists of two constituents, satisfying two properties. To specify $\mathcal{X}$,

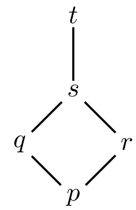
(i) one specifies a set $Ob(\mathcal{X})$, elements which are called *objects*;
(ii) for every two objects x, y, one specifies an element $\mathcal{X}(x, y) \in V$, called the *hom-object*.

The above constituents are required to satisfy two properties:

(a) for every object $x \in Ob(\mathcal{X})$ we have $I \leq \mathcal{X}(x, x)$, and
(b) for every three objects, $x, y, z \in Ob(\mathcal{X})$, we have $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leq \mathcal{X}(x, z)$.

We call $\mathcal{V}$ the *base of the enrichment* for $\mathcal{X}$ or say that $\mathcal{X}$ is enriched in $\mathcal{V}$.

Example 4.9 ([[1], Ex. 2.4.7.])**.** Consider the preorder $(V, \leq)$, given by the following Hasse diagram:



Recall the symmetric monoidal preorder **Bool**=($\mathbb{B}, \leq, \texttt{true}, \wedge$). Let $Ob(\mathcal{X}) = \{p, q, r, s, t\}$ be all the elements contained in the preorder V. We define our hom-object as follows, $\mathcal{X}(x, y) = \texttt{true}$ if $x \leq y$ and $\texttt{false}$ otherwise. With this construction, we can easily check that this indeed is a **Bool-**category.

Our choices may seem very generalized and that is for good reason! In fact, every preorder can be enriched in **Bool** which we will now prove.

Theorem 4.10 ([[1], Thm. 2.49.])**.** *There is a one-to-one correspondence between preorders and* **Bool-***categories.*

Proof. We first wish to show that from any preorder $(P, \leq)$, we can obtain a corresponding **Bool-**category $\mathcal{X}$. We choose the objects of $\mathcal{X}$ as follows,

$$Ob(\mathcal{X}) = \{x \mid x \in P\}.$$

Next, we define the hom-object,

$$\mathcal{X}(x, y) = \texttt{true} \text{ if } x \leq y \text{ and } \texttt{false} \text{ otherwise.}$$

Now we show that constituent (a) holds. Let $x \in Ob(\mathcal{X})$. By reflexivity, $x \leq x$ and so $\mathcal{X}(x, x) = \texttt{true}$. Hence, $I = \texttt{true} \leq \mathcal{X}(x, x)$ as desired. Now we show that constituent (b) holds. Let $x, y, z \in Ob(\mathcal{X})$. Suppose that $\mathcal{X}(x, y) \wedge \mathcal{X}(y, z) = \texttt{true}$. We have that

$$x \leq y \text{ and } y \leq z$$

and so by transitivity $x \leq z$. Hence, $\mathcal{X}(x, z) = \texttt{true}$ which gives us,

$$\mathcal{X}(x, y) \wedge \mathcal{X}(y, z) \leq \mathcal{X}(x, z).$$

If we suppose instead that $\mathcal{X}(x, y) \wedge \mathcal{X}(y, z) = \texttt{false}$, then we have immediately that, $\mathcal{X}(x, y) \wedge \mathcal{X}(y, z) \leq X(x, z)$ since $\texttt{false} \leq \texttt{true}$.

Now, we want to show that from this **Bool-**category, we can obtain the same preorder $(P, \leq)$. Let $P = Ob(\mathcal{X})$. For every $x, y \in P$, we say that $x \leq y$ if and only if $\mathcal{X}(x, y) = \texttt{true}$. Now we check that this preordering is reflexive. Let $x \in P$. We have that $I = \texttt{true} \leq \mathcal{X}(x, x)$. Hence, $\mathcal{X}(x, x) = \texttt{true}$ and so $x \leq x$. Now we check that this preordering is transitive. Let $x, y, z \in P$. Suppose $x \leq y$ and $y \leq z$. We have that $\mathcal{X}(x, y) \wedge \mathcal{X}(y, z) \leq \mathcal{X}(x, z)$. We also have that $\mathcal{X}(x, y), \mathcal{X}(y, z) = \texttt{true}$. Thus, $\mathcal{X}(x, z) = \texttt{true}$ and so $x \leq z$ as desired. □

Remark 4.11. Recall from (4.3) that **Bool** itself is also a preorder. Hence, it is possible to construct a **Bool-**category where the underlying set of objects is the booleans $\mathbb{B}$. It is in this sense that we can consider **Bool** to be *self-enriched*.

The significance of being self-enriched lies in the *closure* of the *hom-object*. That is, given any two objects, $x, y, \in Ob(\mathcal{X})$, the hom-object $\mathcal{X}(x, y) \in Ob(\mathcal{X})$ is also an object. We will formalize this property of *self-enrichment* shortly and discuss a remarkable result which links a certain subset of *self-enriched* monoidal preorders to Galois connections.

Something most readers will be familiar with is the concept of a *metric space*. In particular, we will see that a more generalized version of a metric space, called a *Lawvere metric space* is actually equivalent to a **Cost-**category.

Definition 4.12. A *metric space* (X, d) consists of:

(i) a set X, elements of which are called *points*, and
(ii) a function $d : X \times X \to \mathbb{R}_{\geq 0}$, where $d(x, y)$ is called the *distance between x and y*.

These constituents must satisfy four properties:

(a) for every $x \in X$, we have $d(x, x) = 0$,
(b) for every $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,
(c) for every $x, y \in X$, we have $d(x, y) = d(y, x)$,
(d) for every $x, y, z \in X$, we have $d(x, y) + d(y, z) \geq d(x, z)$.

Definition 4.13. A *Lawvere metric space* is a *metric space* allowing for infinite distances with omission of properties (b) and (c).

One can think of a Lawvere metric space as modelling *cost values* between locations with infinite distances representing impossible paths. Suppose we decide to use transit time by car as our cost value. It is possible that the travel time from the grocery store to our home is not equal to the travel time from our home to the grocery store:

$$d(Store, Home) \neq d(Home, Store).$$

After all, there may be one-way streets which increase the distance and subsequent time taken to travel in one direction. Furthermore, if we consider the travel time between Earth and Mars, we will find that this value is actually infinite (simply because it is physically impossible to reach Mars from Earth solely on car):

$$d(Earth, Mars) = \infty.$$

Proposition 4.14 ([[1], Def. 2.5.3.])**.** *A Lawvere metric space is equivalent to a* **Cost-***category.*

Proof. A **Cost-**category $\mathcal{X}$ consists of:

(i) a set $Ob(\mathcal{X})$,
(ii) for every $x, y \in Ob(\mathcal{X})$ an element $\mathcal{X}(x, y) \in [0, \infty]$.

Here, the set $Ob(\mathcal{X})$ is the set of points, and $X(x, y) \in [0, \infty]$ is distance:

$$X := Ob(\mathcal{X}) \quad d(x, y) := X(x, y).$$

The properties of a category enriched in **Cost** are:

(a) $0 \geq d(x, x)$ for all $x \in X$, and
(b) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Since $d(x, x) \in [0, \infty]$, if $0 \geq d(x, x)$ then $d(x, x) = 0$. Hence, this satisfies (a) of our metric space definition. From (b), we immediately obtain the triangle inequality found in (d) of the metric space properties. We have shown that a **Cost-**category is a Lawvere metric space.

One can now show that from this Lawvere metric space we can obtain the original **Cost-**category that we started with in a similar fashion. □

Recall our discussion in (4.11) about *self-enrichment* with the symmetric monoidal preorder **Bool**. Let us first define what it means to be *self-enriched*.

Definition 4.15. A symmetric monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ is *self-enriched* if there exists a $\mathcal{V}$-category $\mathcal{X}$ where $Ob(\mathcal{X}) = V$.

Example 4.16. See (4.11) which discusses the self-enrichment of **Bool**.

We now wish to introduce a special type of symmetric monoidal preorder. We will call these symmetric monoidal preorders, *closed symmetric monoidal* and we will see that these preorders are indeed enriched in themselves.

Definition 4.17. A symmetric monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ is called *closed symmetric monoidal* if, for every two elements $v, w \in V$, there is an element $v \multimap w \in V$ called the *hom-element*, with the property

$$(a \otimes v) \leq w \text{ if and only if } a \leq (v \multimap w)$$

for all $a, v, w \in V$.

Proposition 4.18. **Bool** *is closed symmetric monoidal.*

Proof. Recall our construction of a **Bool**-category from (4.9). Let $\mathcal{X}$ be a **Bool**-category. We now define the hom-element to be:

$$v \multimap w := \mathcal{X}(v, w) \text{ for every } v, w \in \mathbb{B}.$$

We now wish to show that

$$a \wedge v \leq w \text{ if and only if } a \leq \mathcal{X}(v, w)$$

for every $a, v, w \in \mathbb{B}$. Suppose that $v = \texttt{false}$. We have that $a \wedge v = \texttt{false} \leq w$ and $\mathcal{X}(v, w) = \texttt{true} \geq a$. If we suppose instead that $v = \texttt{true}$, then we have that $a \wedge v = a \leq w$. We also have that $\mathcal{X}(v, w) = w \geq a$. Therefore, **Bool** is closed symmetric monoidal. ☐

From the previous proposition, one may realize that our property for closure looks identical to that of a Galois connection. In particular, we are given $(- \otimes v) : V \to V$ and $(v \multimap -) : V \to V$ as our adjoint functions. We will now prove that this is indeed the case.

Recall (2.15) which showed an alternative definition of a Galois connection. We will now show that this is also the case for a symmetric monoidal closure.

Lemma 4.19. *Suppose $\mathcal{V} = (V, \leq, I, \otimes)$ is closed symmetric monoidal. Then we have that*

$$((v \multimap w) \otimes v) \leq w \text{ and } w \leq (v \multimap (w \otimes v))$$

for all $v, w \in V$.

Proof. Let $a = v \multimap w$. By reflexivity we have that $a \leq a$ and so from our closed symmetric monoidal property we are given that $(a \otimes v) = ((v \multimap w) \otimes v) \leq w$. The proof for the other inequality follows analogously. ☐

Proposition 4.20. *A symmetric monoidal preorder is closed symmetric monoidal if and only if, given any $v \in V$, the maps $(- \otimes v) : V \to V$ and $(v \multimap -) : V \to V$ form a Galois connection with $(- \otimes v)$ and $(v \multimap -)$ as the left and right adjoints respectively.*

Proof. From (4.17), it immediately follows that $(- \otimes v)$ and $(v \multimap -)$ form a Galois connection as the left and right adjoints respectively. Hence, it suffices to show that given a symmetric monoidal preorder which is closed, we have that $(- \otimes v)$ and $(v \multimap -)$ are monotone.

Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a symmetric monoidal preorder which is closed symmetric monoidal. Fix some $v \in V$. Let $a, b \in V$ such that $a \leq b$. From (a) of (4.1), we are given that $a \otimes v \leq b \otimes v$. Hence, $(- \otimes v)$ is monotone. From (4.19), we have that $(v \multimap a) \otimes v \leq a \leq b$. Therefore, from the property of symmetric monoidal closure, it follows that $v \multimap a \leq v \multimap b$; that is, $(v \multimap -)$ is monotone. □

Proposition 4.21. *A closed symmetric monoidal preorder is self-enriched.*

Proof. Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a closed symmetric monoidal preorder. We define our objects to be elements of V and our hom-object to be our hom-element.

$$Ob(\mathcal{X}) = V \qquad \mathcal{X}(x, y) := x \multimap y \text{ for all } x, y \in V$$

Let $x \in V$. From our symmetric monoidal structure we have that $I \otimes x \leq x$. Thus, using our monoidal closure property, we are given $I \leq x \multimap x = \mathcal{X}(x, x)$. Hence, this satisfies (a) from (4.8). Let $u, v, w \in V$. By (4.19) we have that $u \otimes (u \multimap v) \leq v$ and $v \otimes (v \multimap w) \leq w$. Hence, by (a) from (4.1) we have that

$$u \otimes (u \multimap v) \otimes (v \multimap w) \leq w.$$

Applying the closed symmetric monoidal property, we obtain

$$(u \multimap v) \otimes (v \multimap w) \leq u \multimap w$$

satisfying (b) from (4.8). Therefore, $\mathcal{V}$ is self-enriched. □

5. Finite Topological Spaces

We will now offer a very brief glimpse into finite topological spaces. As this paper's primary focus is on symmetric monoidal preorders, we will almost exclusively discuss results which relate to preorders. Readers are encouraged to explore much deeper into the world of finite spaces on their own time (see May [2]) as much is omitted here resulting in frankly lackluster "world-building". As such, it may be helpful for readers to have some background in basic point-set topology prior to reading this section.

Definition 5.1. A *topology* τ on a set X is a set containing subsets of X, called the *open sets of X in the topology τ*, with the following properties:

(i) The empty set $\emptyset$ and the set X are open.
(ii) Finite intersections of open sets are open.
(iii) Arbitrary unions of open sets are open.

We denote a set X with a topology τ as (X, τ). When the topology τ is obvious from context, we say that X is a *topological space.*

Definition 5.2. An *Alexandroff space* X is a topological space such that arbitrary intersections of open sets are open.

Proposition 5.3. *A finite space is an Alexandroff space.*

Proof. In a finite topological space, all arbitrary intersections are finite. □

Although we will only refer to finite topological spaces through our examples, it is important to note that most of our results do generalize to Alexandroff spaces. As such, when possible, our results will be in the context of Alexandroff spaces.

Definition 5.4. A *basis* for a topology on a set X is a set $\mathcal{B}$ containing subsets of X such that

(i) For every $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$.
(ii) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

The topology τ *generated* by a basis $\mathcal{B}$ is the set containing subsets $U \subseteq X$ such that, for every point $x \in X$ there exists some basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Remark 5.5. An equivalent way to generate a topology τ from a basis $\mathcal{B}$ is to simply take $\tau := \{\bigcup\{B_\lambda \mid B_\lambda \in \mathcal{B}\}\}$; that is, our topology is given precisely by all possible unions of basis elements and so every open set is a union of basis elements.

Definition 5.6. Let (X, τ) be a topological space. A *neighborhood* of a point $x \in X$ is an open set $U \in \tau$ containing x.

Definition 5.7. Let X be an Alexandroff space. For each point $x \in X$, we define U_x to be the intersection of all neighborhoods of x. We define a relation $\leq$ on the set X by $x \leq y$ if $U_x \subseteq U_y$. We say that $x < y$ if $U_x \subset U_y$ is a proper subset.

Proposition 5.8. *Let X be an Alexandroff space. The set of all open sets U_x is a basis $\mathcal{B}$ for X. If $\mathcal{C}$ is another basis, then $\mathcal{B} \subseteq \mathcal{C}$, therefore $\mathcal{B}$ is the unique minimal basis for X.*

Proof. For details on this proof see May [[2], Lem. 1.3.6.]. □

Proposition 5.9. *Let X be an Alexandroff space. The ordering $\leq$ on X given by (5.7) is a preordering.*

Proof. Let $x, y, z \in X$. Immediately we have that $U_x = U_x$ and so $x \leq x$, satisfying reflexivity. Suppose $x \leq y$ and $y \leq z$. We have that $U_x \subseteq U_y \subseteq U_z$. Therefore, $U_x \subseteq U_z$ and so $x \leq z$, satisfying transitivity. Hence, $\leq$ is a preordering. □

Lemma 5.10. *Let X be an Alexandroff space. Let $x, y \in X$ be two points.*

$$x \in U_y \text{ if and only if } U_x \subseteq U_y$$

Proof. Suppose $x \in U_y$. Since U_y is an open set, it follows that U_y is a neighborhood of x. Therefore, $U_x \subseteq U_y$ since U_x is the intersection of all neighborhoods of x.

Suppose now that $U_x \subseteq U_y$. We have that $x \in U_x \subseteq U_y$ and so $x \in U_y$. □

Definition 5.11. Let X and Y be spaces. A map $f : X \to Y$ is *continuous* if the pre-image $f^{-1}(U)$ is open in X for every open set $U \in Y$.

Proposition 5.12. *A preorder $(X, \leq)$ determines a topology τ on X with basis $\mathcal{B} = \{U_x \mid x \in X\}$. We call this topology the order topology on X. The space (X, τ) is an Alexandroff space.*

Proof. We first wish to show that $\mathcal{B}$ is a basis for a topology. Let X be a topological space with topology τ. Let $x \in X$. By definition, we have that $x \in U_x$. Hence, (i) from (5.4) is satisfied. Let $y, z \in X$ and suppose $x \in U_y \cap U_z$. Since $x \in U_y$ and

$x \in U_z$, it follows from (5.10) that $U_x \subseteq U_y$ and $U_x \subseteq U_z$. Hence, $x \in U_x \subseteq U_y \cap U_z$ and so (ii) from (5.4) is met. Therefore, $\mathcal{B}$ is a basis for a topology τ.

Now we wish to show that (X, τ) is an Alexandroff space; that is, arbitrary intersections of open sets are open. Consider the set $S = \bigcap\{U_\lambda \mid U_\lambda \in \tau\}$. Let $x \in S$. It follows that $x \in U_\lambda$ for every λ. Hence, $U_x \subseteq U_\lambda$ for every λ and so $x \in U_x \subseteq S$. Thus, S is open which gives us that (x, τ) is an Alexandroff space. □

Proposition 5.13. *Given any set X, there is a one-to-one correspondence between the Alexandroff topologies on X and the preorders on X.*

Proof. For details on this proof see May [[2], Prop. 1.6.4]. □

The importance of this proposition cannot be stressed enough. With knowledge of a one-to-one correspondence between finite topological spaces and finite preorders, this allows us to view preorders in two distinct ways simultaneously. From this, we will be able to connect our results from previous sections and apply them in the context of topology.

Definition 5.14. A continuous bijection $f : X \to Y$ between topological spaces is a *homeomorphism* if its inverse f^{-1} is also continuous. If there exists a homeomorphism between two topological spaces X and Y, we say that X and Y are *homeomorphic*, denoted as $X \cong Y$.

We can consider a homeomorphism in topology to be equivalent to that of an isomorphism in algebra. That is, if two spaces are homeomorphic, then they are equivalent.

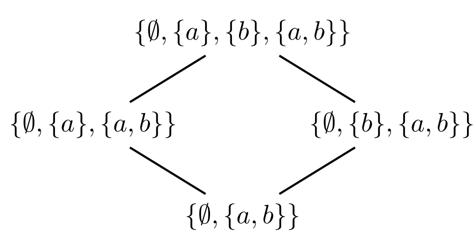
Corollary 5.15. *We have a bijection between finite topologies and Hasse diagrams, so that homeomorphism of spaces is equivalent to graph isomorphism of the two diagrams.*

Proof. This is clear from (2.6) and (5.13). □

Recall from (3.3) our preordering $(\mathbf{Pos}(S), \subseteq)$. It follows then that there is also a level shift present with topological spaces where we can observe a topology of topologies.

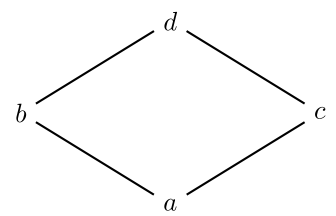
Definition 5.16. Given any set S, we define $\mathbf{Top}(S)$ to be the set of all possible topologies on S.

Example 5.17. Let $X = \{a, b\}$, $Y = \{a, b, c, d\}$. See the Hasse diagram produced by $(\mathbf{Top}(X), \subseteq)$:



.

This Hasse diagram is isomorphic to the Hasse diagram produced by $(Y, \leq)$:



Hence, the preordering $\subseteq$ on $\mathbf{Top}(X)$ is isomorphic to the preordering $\leq$ on Y:

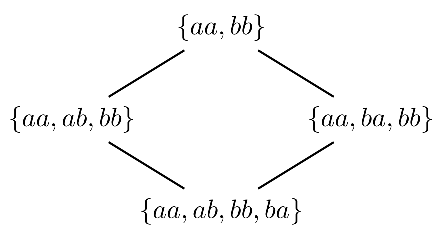
$$\{aa, ab, ac, ad, bb, bd, cc, cd, dd\}.$$

As such, this corresponds to the following topology on a four point set:

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\}.$$

From the preordering on all topologies of a two point set $(\mathbf{Top}(X), \subseteq)$, we were able to generate a new topology $\tau \in \mathbf{Top}(Y)$ on a four point set. Therein lies the level shift.

Remark 5.18. Upon comparing the Hasse diagram from (5.17) with the one from (3.4) which was produced from $(\mathbf{Pos}(X), \subseteq)$, we may notice that the Hasse diagram for $\mathbf{Top}(X)$ corresponds identically to the Hasse diagram of the opposite preordering on $\mathbf{Pos}(X)$. In particular, if we convert the topologies contained in the Hasse diagram above to their corresponding preorders, then the Hasse diagram produced corresponds to $(\mathbf{Pos}(X), \subseteq^{op})$.



Recall (4.7). It follows then that regardless of our choice of subset inclusion with **Top** or with **Pos**, our preordering will always possess a symmetric monoidal structure.

Corollary 5.19. *Given any finite set X, it follows that $(\mathbf{Top}(X), \subseteq, I, \wedge)$ is a symmetric monoidal preorder.*

Proof. Since there is a one-to-one correspondence between topologies and preorders on the set X, this follows directly from (4.6). $\square$

Proposition 5.20. *Given Alexandroff spaces X and Y, a map $f : X \to Y$ is continuous if and only if f is monotone.*

Proof. We will first show the forward direction. Suppose $f : X \to Y$ is continuous. Let $w \leq x$. We want to show that $f(w) \leq f(x)$; that is, $U_{f(w)} \subseteq U_{f(x)}$. We have that $f^{-1}(U_{f(x)}) = \{y \in x \mid f(y) \in U_f(x)\}$. Since $f(x) \in U_{f(x)}$, it follows that $x \in f^{-1}(U_{f(x)})$. Since f is continuous, we also have that $f^{-1}(U_{f(x)})$ is open. Therefore, $f^{-1}(U_{f(x)})$ is a neighborhood of x and so $U_x \subseteq f^{-1}(U_{f(x)})$. Since $w \in U_w \subseteq U_x$, this gives us that $w \in f^{-1}(U_{f(x)})$. Thus, $f(w) \in U_{f(x)}$ and so $U_{f(x)}$ is a neighborhood of $f(w)$. We obtain $U_{f(w)} \subseteq U_{f(x)}$; that is, $f(w) \leq f(x)$ as desired. Therefore, f is monotone.

We will now show the reverse direction. Suppose f is monotone. Let $V \subseteq Y$ be open in Y. Let $x \in X$ such that $f(x) \in V$. We have that $f(x) \in U_{f(x)} \subseteq V$ and so $x \in f^{-1}(V)$. Let $w \in U_x$. It follows that $U_w \subseteq U_x$ and so $w \leq x$ which gives us that $f(w) \leq f(x)$ since f is monotone. Thus, we have that $f(w) \in U_{f(w)} \subseteq U_{f(x)} \subset V$, and so $w \in f^{-1}(V)$. Hence, $U_x \subset f^{-1}(V)$. Therefore, for every $x \in f^{-1}(V)$, we have that $x \in U_x \subset f^{-1}(V)$ and so $f^{-1}(V)$ is open in X; that is, f is continuous. □

Definition 5.21. Let $I = [0, 1]$ denote the unit interval as a topological space with its standard metric topology as a subspace of the real numbers $\mathbb{R}$.

Definition 5.22. Let X and Y be topological spaces. A *homotopy* h is a continuous map $h : X \times I \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. We say that two maps are *homotopic*, written $f \simeq g$, if there exists a homotopy between them.

Definition 5.23. Let X and Y be topological spaces. X and Y are *homotopy equivalent* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. We call f and g *homotopy inverses* and we call the pair of maps (f, g) a *homotopy equivalence.*

Recall the equivalent definition of a Galois connection which we obtained in (2.15). If we compare that to our definition of a homotopy equivalence we may notice the following:

$$
\begin{aligned}
p \leq g(f(p)) \quad &\approx \quad id_X \simeq g \circ f \\
f(g(q)) \leq q \quad &\approx \quad f \circ g \simeq id_Y
\end{aligned}
$$

where the left/right side displays our conditions for a Galois connection/homotopy equivalence. Therefore, the following proposition may not come as a surprise.

Proposition 5.24. *If f and g form a Galois connection then f and g are homotopy inverses of each other.*

Proof. For details on this proof see Ayzenberg [[3], Cor. 3.1.]. □

Corollary 5.25. *Given a symmetric monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ which is closed symmetric monoidal, we have that $(- \otimes v)$ and $(v \multimap -)$ are homotopy inverses of each other.*

Proof. Recall from (4.20) that $(- \otimes v)$ and $(v \multimap -)$ form a Galois connection. Thus, by (5.24) they are homotopy inverses. □

Corollary 5.26. *Given any finite set S, $\mathbf{Rel}(S)$ and $\mathbf{Pos}(S)$ are homotopy equivalent.*

Proof. Recall from (3.5) that there exists a Galois connection between $\mathbf{Rel}(S)$ and $\mathbf{Pos}(S)$. Thus, by (5.24) they are homotopy equivalent. □

Corollary 5.27. *Let S be a finite set. Let (X, τ_T) be the topological space corresponding to $(\mathbf{Top}(S), \subseteq)$ and let (Y, τ_R) be the corresponding topological space to $(\mathbf{Rel}(S), \subseteq)$. There is a homotopy equivalence between X and Y.*

Proof. This follows immediately from (5.26). □

Acknowledgments

Firstly, I would like to thank my mentor, Adolfo Medina, for guiding me along this very unfamiliar process. Without his assistance, this paper would not have been possible. I especially am grateful for his willingness in allowing me free exploration into various rabbit holes of mathematics (most which did not make it into this paper) throughout this program. Next, I would like to thank Carmen Rovi, for encouraging me to explore the wonderful world of symmetric monoidal preorders in the first place, a decision which essentially birthed this paper. I also would like to thank Peter May for his organization of the program and his finite space lectures which motivated me to read more topology on my own time. Last but certainly not least, I would like to thank my friend, Nico Marin Gamboa, for the late nights spent in aimless discussion on mathematics, his recommendation of a text which proved instrumental in the creation of this paper, and for his willingness to deal with my onslaught of questions, stemming from my inexperience.

References

[1] B. Fong, D. I. Spivak. Seven Sketches in Compositionality: An Invitation to Applied Category Theory. Cambridge University Press. 2019.

[2] J. P. May. Finite Spaces and Larger Contexts. Available at http://www.math.uchicago.edu/~may/REU2023/FiniteSpacesNew

[3] A. Ayzenberg. Topology of nerves and formal concepts. Available at https://arxiv.org/pdf/2307.01040.pdf

[4] J. W. Walker. Homotopy Type and Euler Characteristic of Partially Ordered Sets. European Journal of Combinatorics, Volume 2, Issue 4, (1981), 373-384.

[5] A. Patel, P. Skraba. Möbius Homology. Available at https://arxiv.org/pdf/2307.01040.pdf

[6] M. L. Wachs. Poset Topology: Tools and Applications. IAS/Park City Mathematics Series Volume 00 (2004).

[7] M. Erdmann. Topology of Privacy: Lattice Structures and Information Bubbles for Inference and Obfuscation. Available at https://www.cs.cmu.edu/~me/papers/TopPrivacy.pdf

[8] E. Cheng, A. Lauda. Higher-Dimensional Categories: an illustrated guide book. Available at https://eugeniacheng.com/wp-content/uploads/2017/02/cheng-lauda-guidebook.pdf

[9] J. R. Munkres. Topology; A First Course. Prentice-Hall College Div. 1974.